Unit -I : Topic - Differential Calculus

Engg. Maths-I

Direction – (Q^n 1 to Q^n 9) : Use 'Maclaurians Theorem' to solve following ?

Q.1) Find the value of A,B & C in the expansion of the function log (secx) = $Ax^2 + Bx^4 + Cx^6 + \ldots$

Q.2) Expand the function f(x) = $log_e(1 + x)$

Q.3) Find first four terms in expansion of function log(1 + sinx

Q.4) Expand the function f(x) = $log_e(1 + e^x)$

Q.5) Expand e^{sinx} upto the term containing x^4

Q.6) Expand $e^x.cosx$

Q.7) Expand $e^x.sinx$

Q.8) Expand $\tan^{-1}x$ in ascending power of x

Q.9) Expand $e^{a\sin^{-1}x}$ by Maclaurian's theorem

Direction – (Q^n 10 to Q^n 18) : Use 'Taylor's Theorem' to solve following ?

Q.10) Expand sinx in power of $(x - \frac{\pi}{2})$

Q.11) Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - ....$

Q.12) Expand $2x^3 + 7x^2 + x - 1$ in power of (x – 2)

Q.13) Expand $2 + x^2 - 3x^5 + 7x^6$ in powers of (x – 1).

Q.14) Find the Taylor's series expansion of the function f(x) = log cosx about the point $\frac{\pi}{3}$

Q.15) Expand $\tan(x + \frac{\pi}{4})$ as far as the term $\mathbf{x^4}$ & evaluate tan46.5° upto four significant fig. digit

Q.16) Expand $\log_e x$ in power of (x – 1) & hence evaluate $\log_e(1.1)$ correct upto four decimal places

Q.17) Compute the approximate value of $\sqrt{11}$ to four decimal places by taking the first five terms as an approximate Taylor's expansion

Q.18) Use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta.\frac{\sin\theta}{1} - (h\sin\theta)^2\frac{\sin 2\theta}{2} + (h\sin\theta)^3\frac{\sin 3\theta}{3} - .... + (-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n} + ...$$

Direction – (Q^n 19 to Q^n 22) :Use partial differentiation in 2 variables to find Maxima & Minima

Q.19) Discuss the maximum or minimum values of the function f(x , y) = $x^3 - 4xy + 2y^2$.

Q.20) Discuss the maxima and minima of the function $x^3 + y^3 - 3axy$.

Q.21) Discuss the maximum or minimum value of u = $x^3y^2(1 - x - y)$

Q.22) Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

Direction – (Q^n 23 to Q^n 25):Use Center of Curvature & Radius of curvature to solve following ?

Q.23) Find radius of curvature of y = 4 sin x – sin 2x at x = $\frac{\pi}{2}$

Q.24) Find the coordinates of the center of curvature for the point (x,y) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Q.25) Find the center of curvature at the point (1,-1) of the curve $y = x^3 - 6x^2 + 3x + 1$. Hence find the equation of the circle of curvature at this point?

LEIBNITZ'S THEOREM: If u and v are any two functions of x such that all their desired differential coefficients exist, then the n^{th} differential coefficient of their product is given by

$$D^n(uv) = (D^n u).v + nD^{n-1}u.Dv + \frac{n(n-1)}{2!}D^{n-2}uD^2v + \dots\dots\dots\dots + nDuD^{n-1}v + uDv.$$

Example.

If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

and $x^2y_{n+2} + (2n-1)xy_{n+1} + (n^2+1)y_n = 0.$

Solution.

Let $y = a\cos(\log x) + b\sin(\log x)$,

$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x)\frac{1}{x}$ or $xy_1 = -a\sin(\log x) + b\cos(\log x)$

Now again differentiating both sides, we get

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

or $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$

or $x^2y_2 + xy_1 = -y$

or $x^2y_2 + xy_1 + y = 0.$

Again differentiating both sides in times by Leibnitz's theorem,

$$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0.$$

or $x^2D^ny_2 + nDx^2D^{n-1}y_2 + \frac{n(n-1)}{2}D^2x^2D^{n-2}y_2 + xD^ny_1 + nD^{n+1}y_1 + y_n = 0$

or $x^2y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

or $x^2y_{n+2} + (2n-1)xy_{n+1} + (n^2+1)y_n = 0.$

Direction – (Q^n 26 to Q^n 28):Solve Using Leibnitz's Theorem ?

Q.26) If $y = \sin(m\sin^{-1}x)$. prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and deduce that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0.$

Q.27) Find y_n if $y = x^{n-1}\log x.$

Hint.

$\therefore xy_1 = (n-1)x^{n-1}\log x + x^{n-1} = (n-1)y + x^{n-1}.$

Differentiating both sides $(n-1)$times, we have

$D^{n-2}(xy_1) = (n-1)D^{n-1}y + D^{n-1}x^{n-1}.$

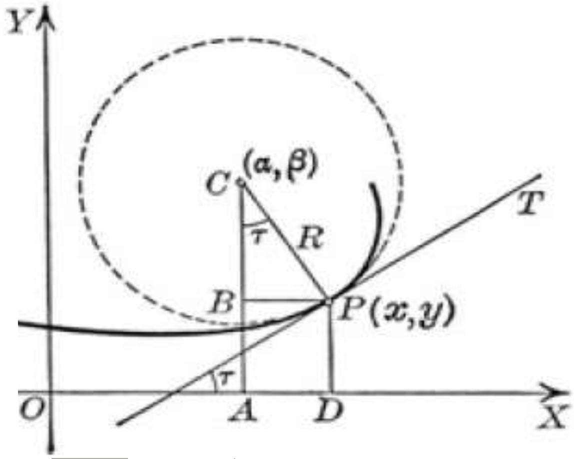
$\therefore xy_n + (n-1)y_{n-1} = (n-1)y_{n+1} + (n-1)!$ or $y_n = \frac{(n-1)}{x}$

Q.28) Find $(y_n)_0$. if $y = \sin(a\sin^{-1}x).$

Radius of curvature & Co-ordinate of Centre of Curvature -

$$R = \pm \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\alpha = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}; \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$



Ex. **Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3 + y^3 = 3axy$.**

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{(ay-x^2)}{(y^2-ax)}} \quad \ldots \quad (1)$$

Here, $\boxed{\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1}$

Again, differentiating (*1*) , w.r.t x , we get

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx}-2x\right)-(ay-x^2)\left(2y\frac{dy}{dx}-a\right)}{(y^2-ax)^2}$$

$$\Rightarrow \boxed{\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{32}{3a}}$$

Now, the Radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is given by $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{[1+(-1)^2]^{\frac{3}{2}}}{\left(-\frac{32}{3a}\right)} = -\frac{2^{\frac{3}{2}}}{32}.3a = -\frac{2\sqrt{2}}{32}3a$$

$$\Rightarrow (\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{3\sqrt{2}}{16}a$$ (numerically... since radius cannot be negative)