Unit -3 : Topic - Integral Calculus

Engg. Mathematics-I

Direction – (Q^n^ 1 to Q^n^ 5) : Use " Integral as limit of sum concept " to solve following ?

Q.1) $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right]$

Q.2) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{8n}\right]$

Q.3) $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+3}+\frac{1}{n+5}+\cdots+\frac{1}{3n-1}\right]$

Q.4) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$

Q.5) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\ldots\left(1+\frac{1}{n}\right)^{1/n}\right]$

Direction – (Q^n^ 6 to Q^n^ 12) : Solve following integral using Beta & Gamma function ?

Q.6) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}dx$

Q.7) $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$

Q.8) Prove that $\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{\log c^{c+1}}$

Q.9) $\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$

Q.10) $\int_0^2 x\sqrt[3]{(8-x^3)}\, dx$

Q.11) State & prove relation between beta & gamma function?

Q.12) State & prove duplication formula?

Direction – (Q^n^ 13 to Q^n^ 26) : Solve following Double & Triple Integration ?

Q.13) $\int_0^1\int_0^x e^{y/x}\, dx\, dy$

Q.14) $\int_0^2\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

Q.15) $\int_0^2\int_0^{x^2} e^{y/x}\, dx\, dy$

Q.16) $\int_0^a\int_0^{\sqrt{ay}} xy\, dx\, dy$

Q.17) $\int_0^a\int_0^{\sqrt{a^2-y^2}} dy\, dx$

Q.18) $\int_0^1\int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{1}{\sqrt{1-x^2-y^2}}\, dx\, dy$

Q.19) $\int_0^3\int_0^2\int_0^1 (x+y+z)dx\, dy\, dz$

Q.20) $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dz\, dy\, dx$

Q.21) $\int_0^2\int_0^x\int_0^{x+y} e^x(y+2z)dx\, dy\, dz$

Q.22) $\int_0^1\int_{y^2}^1\int_0^{1-z} z\, dy\, dz\, dx$

Q.23) $\int_1^3\int_{1/x}^1\int_0^{\sqrt{xy}} xyz\, dx\, dy\, dz$

Q.24) $\int_0^a\int_0^{b\sqrt{\left(1-x^2/a^2\right)}}\int_0^{c\sqrt{\left(1-x^2/a^2-y^2/b^2\right)}} dx\, dy\, dz$

Q.25) $\int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$

Q.26) $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\, dy\, dx\, dz$

Direction – (Q^n^ 27 to Q^n^ 32) : Solve following Double Integration by using change of order concept?

Q.27) $\int_0^3\int_0^{3-x} xy\, dy\, dx$

Q.28) $\int_0^1\int_{x^2}^{2-x} xy\, dy\, dx$

Q.29) $\int_1^2\int_0^x \frac{1}{x^2+y^2}\, dx\, dy$

Q.30) $\int_0^2\int_x^{3x-x^2}(3x^2-2xy)dx\, dy$

Q.31) $\int_0^3\int_0^{3-x} x\, dy\, dx$

Q.32) $\int_0^1\int_{x^2}^{2-x} x\, dy\, dx$

Gamma & Beta f^n

Gamma f^n :-

Gamma f^n denoted as Γn is a definite $\int$ defined as

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx}$$

Some properties of Γ f^n

① $\Gamma 1 = 1$

② $\Gamma n+1 = n\Gamma n$ $\forall$ n being fraction

$= n!$ $\forall$ n being z^+

③ $\dfrac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

④ $\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$ $\forall \alpha < 1$

⑤ $\Gamma m \, \Gamma m+\frac{1}{2} = \dfrac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$

⑥ $\Gamma \frac{1}{2} = \sqrt{\pi}$

⑦ $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \dfrac{\Gamma m \, \Gamma n}{2\Gamma m+n}$

Proof of Legendre Duplication formula

To prove :- $\Gamma m \, \Gamma m+\frac{1}{2} = \dfrac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$ ①

Proof :- $\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \dfrac{\Gamma m \Gamma n}{2\Gamma m+n}$

put $2n-1=0 \Rightarrow n=\frac{1}{2}$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \sqrt{\pi}}{2\Gamma m+\frac{1}{2}} \quad (II)$$

now put $n=m$ in ① we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \Gamma m}{2\Gamma 2m}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\, 2\, d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

let $2\theta = \phi \Rightarrow 2d\theta = d\phi$

$$\frac{1}{2^{2m}}\cdot\int_0^{\pi} (\sin\phi)^{2m-1} d\phi = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$\phi \leftrightarrow \theta$

$$\frac{2}{2^{2m}}\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = 2^{2m-1}\frac{(\Gamma m)^2}{2\Gamma 2m} \quad (III)$$

from (II) & (III)

$$\frac{\Gamma m \sqrt{\pi}}{\cancel{2}\,\Gamma m+\frac{1}{2}} = \frac{2^{2m-1}(\Gamma m)^2}{\cancel{2}\,\Gamma 2m}$$

$$\boxed{\Gamma m \, \Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}}$$

Beta function :-

Beta f^n is denoted by $B(m,n)$, is also a def. $\int$

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx}$$

$m, n \in R^+$

Some properties of B(m,n) f^n

① $B(m,n) = B(n,m)$

(2) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

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(21) $\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(5) $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Proof of Relation b/w β & Γ function

To prove:—

$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Proof:-

we know that

$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1} dx$

$\Rightarrow \Gamma n = \int_0^\infty e^{-zx} x^{n-1} z^n dx$

multiplying both side by $e^{-z} z^{m-1}$ we get

$\Gamma n \, e^{-z} z^{m-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1} dx$

Integrating both side w.r.t z

$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] dx$

$\Gamma n \, \Gamma m = \int_0^\infty x^{n-1} \frac{\Gamma m+n}{(1+x)^{m+n}} dx$

$\boxed{\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)}$

Assignment Questions

Q.(2) a) $\int_0^\infty \sqrt[4]{x} \, e^{-\sqrt{x}} dx$

Sol^n. put $x = t^2 \Rightarrow dx = 2t\,dt$

$= \int_0^\infty x^{1/4} e^{-x^{1/2}} dx$

$= \int_0^\infty t^{1/2} e^{-t} 2t\,dt$

$= 2\int_0^\infty e^{-t} t^{1+1/2} dt$

$= 2\int_0^\infty e^{-t} t^{5/2-1} dt$

$= 2 \times \Gamma 5/2$

$= 2 \times 3/2 \times 1/2 \times \Gamma 1/2$

$= \frac{3}{2}\sqrt{\pi}$

b) $\int_0^\infty \sqrt{x} \, e^{-\sqrt[3]{x}} dx$

Sol^n. $x = t^3 \Rightarrow dx = 3t^2 dt$

$\int_0^\infty t^{3/2} e^{-t} 3t^2 dt$

$= 3\int_0^\infty t^{7/2} e^{-t} dt$

$= 3 \times 7/2 \times 5/2 \times 3/2 \times 1/2 \, \Gamma 1/2$

$= \frac{315}{16}\sqrt{\pi}$

c) $\int_0^2 x \sqrt[3]{8 - x^3} dx$

Sol^n. $\int_0^2 x (8 - x^3)^{1/3}$

let $x = 2t^{1/3}$

$dx = \frac{2}{3} t^{-2/3} dt$

$= \int_0^1 2t^{1/3} (8 - 8t)^{1/3} \frac{2}{3} t^{-2/3} dt$

$= \frac{4 \times 2}{3} \int_0^1 t^{-1/3} (1-t)^{1/3} dt$

$= 8/3 \int_0^1 t^{\frac{2}{3}-1} (1-t)^{\frac{4}{3}-1} dt$

$= 8/3 \, B\left(\frac{2}{3}, \frac{4}{3}\right)$