Unit - I , Topic - Fourier Series & Fourier Transform

PERIODIC FUNCTIONS - The function f(x) defined for all *real values* of *x* is said to be a **periodic function** if it repeats itself after equal interval of time say T such that $f(x) = f(x+T) = f(x+2T) = ...$ for all x

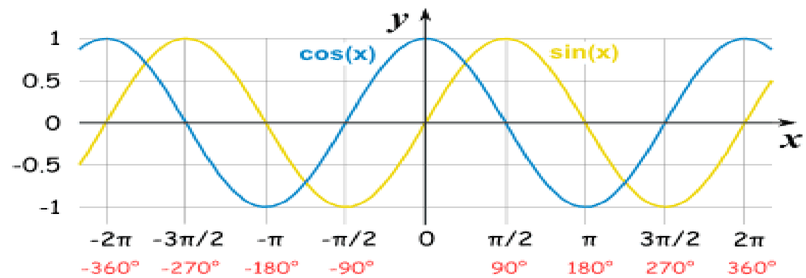
Here the *smallest positive value* of T is called the **primitive period** or **least period** or simply the **period** of f(x).

For example:- f(x) = sin x or cos x have period 2π

since sin x = sin (x + 2π) = sin (x + 4π) = …….

similarly cos x = cos (x + 2π) = cos (x + 4π) = …….

The behavior of these **sinusoidal periodic functions** (sin x & cos x) is shown in graphs.



EVEN FUNCTIONS – A function f(x) is an even function if $f(-x) = f(x)$

For Example $f(x) = \cos x$ gives $f(-x) = \cos(-x) = \cos x = f(x)$

ODD FUNCTIONS – A function f(x) is odd function if $f(-x) = -f(x)$

For Example $f(x) = \sin x$ gives $f(-x) = -\sin x = -f(x)$

PROPERTIES OF EVEN & ODD FUNCTIONS

1. Multiplication of two odd function gives even function for example x sin x
2. Multiplication of even and odd function gives odd function for example x cos x
3. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, **if f(x) is even function**
4. $\int_{-a}^{a} f(x)dx = 0$, **if f(x) is odd function**
5. The property of even and odd function is used only when the limit of integration is **– a** to **+ a**

SOME USEFUL FORMULAE

1. $\sin n\pi = 0$
2. $\cos n\pi = (-1)^n$
3. $\cos 2n\pi = 1$
4. $\cos(2n-1)\frac{\pi}{2} = 0$
5. $\sin x \cos nx = \frac{1}{2}[\sin(n+1)x - \sin(n-1)x]$
6. $\sin x \sin nx = \frac{1}{2}[\cos(n-1)x - \cos(n+1)x]$
7. $\sin n\pi = \cos\left(n+\frac{1}{2}\right)\pi = 0$
8. $\cos n\pi = \sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$
9. $\int \sin ax\, dx = \frac{-\cos ax}{a}$
10. $\int \cos ax\, dx = \frac{\sin ax}{a}$
11. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
12. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

13. Generalized rule of integration by parts (**Bernoulli's formula**).

If u and v are function of x, then $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$

Where u', u",… are successive derivatives of u and v_1, v_2,… are successive indefinite integrals of v. This formula will be useful if u is a polynomial.

For example :- $\int x^2 \sin nx\,dx = \left[x^2\left(\frac{-\cos nx}{x}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]$

FOURIER SERIES

Any function f(x) which is defined in interval (k, k+ 2*l*) can be expanded in terms of infinite series having sine and cosine terms if f(x) satisfies Dirichelet's conditions.

DIRICHELET'S CONDITIONS

1. f(x) should be a periodic function
2. f(x) should be single valued in the interval (k, k + 2*l*)
3. Function should have finite number of maxima and minima in the interval (k, k + 2*l*)
4. f(x) should have finite number of discontinuity in the interval (k, k+ 2*l*)

Then, at any point of continuity f(x) can be expanded as

$$f(x) = \frac{a_0}{2} + a_1\cos\left(\frac{x\pi}{l}\right) + a_2\cos\left(\frac{2x\pi}{l}\right) + a_3\cos\left(\frac{3x\pi}{l}\right) + ... + a_n\cos\left(\frac{nx\pi}{l}\right) + ...$$

$$+ b_1\sin\left(\frac{x\pi}{l}\right) + b_2\sin\left(\frac{2x\pi}{l}\right) + b_3\sin\left(\frac{3x\pi}{l}\right) + ... + b_n\sin\left(\frac{nx\pi}{l}\right) + ...$$

OR

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

Here a_0, a_n, b_n are constant called Fourier constants / Coefficients and can be determined by Euler's formula as follow

$$a_0 = \frac{1}{l}\int_k^{k+2l} f(x)dx\ ,\quad a_n = \frac{1}{l}\int_k^{k+2l} f(x)\cos\frac{n\pi x}{l}dx\ ,\quad b_n = \frac{1}{l}\int_k^{k+2l} f(x)\sin\frac{n\pi x}{l}dx$$

HALF RANGE FOURIER SERIES

A function f(x) defined over the interval $0 \le x \le l$ can be expanded in two distinct half- range series.

(1) The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$, n = 1, 2, 3, ….

(2) The half-range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$, n = 1, 2, 3, ….