Unit – II , Topic – Laplace Transform

It is the work of a French Mathamatician dierre – de – Laplace (1749 – 1827). This method reduces the problem of differential equation with given boundary condition to simple Algebraic problem.

Definition

Let F(t) be a function of 't' defined for all positive value of t; then the Laplace transform F(t) is denoted by

$L\{F(t)\}$, $f(s)$ or $\overline{F}(s)$ and is defined as $L\{F(t)\}=\int_0^\infty e^{-st}F(t)dt = f(s)$

Here S is a parameter may be real or complex provided that integral is convergent.

Laplace Transformation of some Elementary Functions

(i) $L\{(1)\}=\frac{1}{S}$

Proof By definition $L\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$ So $L\{1\}=\int_0^\infty e^{-st}1dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=-\frac{1}{S}\left[\frac{1}{e^{st}}\right]_0^\infty=-\frac{1}{S}\left[\frac{1}{\infty}-1\right]=-\frac{1}{S}[0-1]=\frac{1}{S}$

(ii) $L\{t^n\}=\frac{\overline{|n+1}}{S^{n+1}}$ if n > - 1; n is non integral / fractional value say $\frac{1}{2},\frac{3}{2},\frac{5}{2}$, ...

(iii) $L\{t^n\}=\frac{\underline{|n}}{S^{n+1}}$ if n is positive integers say 1,2,3….

(iv) $L\{e^{at}\}=\frac{1}{S-a}$, (for S > a) Corollary - $L(c^{at})=L\left[e^{at\log c}\right]=\frac{1}{s-a\log c}$ if s > a log c, c > 0

(v) $L\{\sin at\}=\frac{a}{S^2+a^2}$ (vi) $L\{\cos at\}=\frac{S}{S^2+a^2}$ (vii) $L\{\sin h\,at\}=\frac{a}{S^2-a^2}$ (viii) $L\{\cos h\,at\}=\frac{S}{S^2-a^2}$

Some Elementary Formulae

1. $\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$
2. $\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$
3. $\sin A\sin B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$
4. $\sin^3 A=\frac{1}{4}[3\sin A-\sin 3A]$
5. $\cos^3 A=\frac{1}{4}[3\cos A+\cos 3A]$
6. $\sin^2 A=\frac{1}{2}[1-\cos 2A]$
7. $\cos^2 A=\frac{1}{2}[1+\cos 2A]$
8. $\sin h\,at=\frac{e^{at}-e^{-at}}{2}$
9. $\cos h\,at=\frac{e^{at}+e^{-at}}{2}$
10. $\sin h^2\,at+\cos h^2\,at=\cos h2at$
11. $\cos h^2\,at-\sin h^2\,at=1$
12. $\overline{|n+1}=\underline{|n}$
13. $\overline{|\tfrac{3}{2}}=\frac{1}{2}\overline{|\tfrac{1}{2}}=\frac{1}{2}\sqrt{\pi}$
14. $\overline{|\tfrac{1}{2}}=\sqrt{\pi}$

Properties of Laplace Transform

1. I-Shifting theorem or I-Translation property

If L{f(t)} = F(s) Then $L\{e^{at} f(t)\} = F(s - a)$

For e.g. – a) $L\{e^{at} sinbt\} = \frac{b}{(s-a)^2+b^2}$ b) $L\{e^{at} cosbt\} = \frac{s-a}{(s-a)^2+b^2}$ c) $L\{e^{at} \sin h\, bt\} = \frac{b}{(s-a)^2-b^2}$

d) $L\{e^{at} \cos h\, bt\} = \frac{s-a}{(s-a)^2-b^2}$ e) $L\{e^{at} t^n\} = \frac{(n+1)!}{(s-a)^{n+1}}$ or $\frac{n!}{(s-a)^{n+1}}$

2. II-Shifting theorem

If L{f(t)} = F(s), then $L[f(t-a).u(t-a)] = e^{-as} F(s)$ i.e. t > a

For e.g. $L\{t^2.u(t-3)\} = e^{-3s} L\{(t+3)^2\} = e^{-3s} L[t^2+6t+9] = e^{3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$

3. Multiplication by 't'

If L[$f(t)$] = F(s), then $L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

For e.g. To find Laplace Transform of $f(t) = t\, e^{-t} \sin 2t$ **[RGPV – June 2004 , Dec.2014]**

Solution. $L[\sin 2t] = \frac{2}{s^2+4}$, $L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4} = F(s)$ (say)

$$L[t\, e^{-t}\sin 2t] = -F'(s) = -\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right] = \frac{4(s+1)}{[(s+1)^2+4]^2}$$

4. Division by 't'

If $L[f(t)] = F(s)$, then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds$

For e.g. To find Laplace Transform of $L\left[e^{-4t}\frac{\sin 3t}{t}\right]$ **[RGPV – 2007 , 12]**

Solution. $L[\sin 3t] = \frac{3}{s^2+3^2} \Rightarrow L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9} ds = \left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3} = \tan^{-1}\frac{3}{s+4}$$

5. Laplace Transform of Derivative of F(t)

If $L[f(t)] = F(s)$ then $L[f'(t)] = sL[f(t)] - f(0)$

In general $L[f^n(t)] = s^n L[f(t) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - ..... - f^{n-1}(0)$

6. Laplace Transform of Integral of F(t)

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s} F(s)$

7. Laplace Transform of periodic Function F(t) periodic with Period T

If $L[f(t)] = F(s)$ then $L[f(t)] = \frac{\int_0^T e^{-st} f(t) dt}{1-e^{-sT}}$

Evaluating Integrations using Laplace Transform –

Q. Evaluate $\int_0^\infty te^{-t} sint\, dt$ **[RGPV – 2005, 06, 13]**

Solution - $\int_0^\infty te^{-3t}\sin t\, dt = \int_0^\infty te^{-st}\sin t\, dt$,{ put s = 3}, $= L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$