



$f(s)$ is the Laplace Transform of $F(t)$ which is denoted as $L[F(t)] = f(s)$ or $F(t) = L^{-1}f(s)$.

$F(t)$ is called as the inverse Laplace Transform of $f(s)$.

Laplace Inverse Transformation of some Elementary Functions

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma n}$ where $n = 0, 1, 2, 3, \dots$
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
5. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a} \sinh at$
6. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$
7. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
8. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \sin bt$
9. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cos bt$
10. $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b} e^{at} \sinh bt$
11. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at} \sinh bt$
12. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3} (\sin at - at \cos at)$
13. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t \sin at$
14. $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t \cos at$
15. $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a} [\sin at + at \cos at]$
16. $L^{-1}\frac{1}{s^2+25} = \frac{1}{5} \frac{5}{s^2+(5)^2} = \frac{1}{5} \sin 5t$

Properties of Inverse Laplace Transform

1. Linearity property – If $L^{-1}\{f_1(s)\} = F(t)$ then $L^{-1}\{af_1(s) \pm bf_2(s)\} = aL^{-1}\{f_1(s)\} \pm bL^{-1}\{f_2(s)\}$

Ex. $L^{-1}\left(\frac{1}{s^4} + \frac{3}{s-2} + \frac{2s}{s^2+9}\right) = L^{-1}\left(\frac{1}{s^4}\right) + 3L^{-1}\left(\frac{1}{s-2}\right) + 2L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{t^3}{3!} + 3e^{2t} + 2\sin 3t$

2. I-Shifting theorem – If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(s-a)\} = e^{at}L^{-1}\{f(s)\} = e^{at}F(t)$

Ex. $\frac{1}{2s-7} = \frac{1}{2} L^{-1}\left(\frac{1}{s-\frac{7}{2}}\right) = \frac{1}{2} e^{\frac{7}{2}t}$

3. II-Shifting theorem – If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{e^{-as}f(s)\} = F(t-a) \quad \forall t > a$

4. Inverse Laplace Transform of Derivative – $L^{-1}\{f'(s)\} = -tF(t)$ in general $L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$

Q. $L^{-1}\left(\tan^{-1}\frac{1}{s}\right) = ?$

Sol. $f(t) = L^{-1}\left(\tan^{-1}\frac{1}{s}\right) \Rightarrow tf(t) = -L^{-1}f'(s) = -L^{-1}\left(\frac{-1}{s^2+1}\right) = \sin t$, Hence $f(t) = \frac{\sin t}{t}$

5. Inverse Laplace Transform of Integral – $L^{-1}\left\{\int_s^\infty f(s)ds\right\} = \frac{F(t)}{t}$

6. Multiplication by 's' – If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{sf(s)\} = F'(t)$ if $f(0)=0$

Q. Find the inverse Laplace transform of $\frac{s}{s^2+1}$

Solution : $L^{-1}\frac{1}{s^2+1} = \sin t \Rightarrow L^{-1}\frac{s}{s^2+1} = \frac{d}{dt}(\sin t) = \cos t$

7. Division by 's' – If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t L^{-1}\{f(s)\}dt$

8. Convolution – If $F(t) = L^{-1}\{f(s)\}$ & $G(t) = L^{-1}\{g(s)\}$ are any two continues function then convolution of F(t) and G(t) is denoted by $F(t)*G(t)$ and defined as $F(t)*G(t) = \int_0^t F(x)G(t-x) dx$

Q. Apply Convolution theorem to evaluate $L^{-1}\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ [J.N.T.U. – 2005, R.G.P.V. – 2004, 2008, 2014]

Solution : Since $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$, $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \cos bt$ $\therefore L^{-1}\left(\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right) = \int_0^t \cos a(t-x) \cos bx dx$
 $= \frac{1}{2} \int_0^t \{\cos[at+(b-a)x] + \cos[at-(b+a)x]\} dx = \frac{1}{2(b^2-a^2)} (2b \sin bt - 2a \sin at) = \frac{1}{b^2-a^2} (b \sin bt - a \sin at)$

Application of Laplace Transform to solve Differential Equations

Working Procedure to solve a linear differential equation with constant coefficients by transform method.

1. Take the Laplace transform of both sides of the differential equation using the formula of and the given initial conditions.
2. Transpose the terms with minus signs to the right.
3. Divide by the coefficient of \bar{y} , getting \bar{y} as a known function of s.
4. Resolve this function of s into partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Q. Solve by the method of transforms, the equation $y'''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$.

Solution : Taking the Laplace transform of both sides, we get [VTU – 2004][RGPV – Dec. 04, Jan 08, Dec13]

$$[s^3 \bar{y} - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2 \bar{y} - s y(0) - y'(0)] - [s \bar{y} - y(0)] - 2 \bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2) \bar{y} = 6 \Rightarrow \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion, we get $y = L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+2}\right) + 2L^{-1}\left(\frac{1}{s+2}\right) \Rightarrow y = e^t - 3e^{-t} + 2e^{-2t}$

Q. Using L.T. , solve $(D-2)x - (D+1)y = 6e^{3t}$ & $(2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3, y = 0$ When $t = 0$.

Solution : We have the following simultaneous equations:

$$(D-2)x - (D+1)y = 6e^{3t} \quad \dots(i)$$

$$(2D-3)x + (D-3)y = 6e^{3t} \quad \dots(ii)$$

Taking the L.T. of both sides of (i) and (ii), we get

$$[s\bar{x} - x(0) - 2\bar{x}] - [s\bar{y} - y(0) + \bar{y}] = \frac{6}{s-3} \quad \dots(iii)$$

$$\text{and } [2\{s\bar{x} - x(0)\} - 3\bar{x}] + [s\bar{y} - y(0) - 3\bar{y}] = \frac{6}{s-3} \quad \dots(iv)$$

using the conditions $x(0) = 3, y(0) = 0$ in (iii) & (iv)

$$(s-2)\bar{x} - (s+1)\bar{y} = \frac{3s-3}{s-3} \text{ and}$$

$$(2s-3)\bar{x} + (s-3)\bar{y} = \frac{6s-12}{s-3}$$

$$\text{Solving these, we get } \bar{x} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{s-3} \quad \dots(v)$$

$$\text{and } \bar{y} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{s-3} \quad \dots(vi)$$

Taking the inverse L.T. of both sides of (v) and (vi)

$$x = e^t + 2te^t + 2e^{3t} \text{ and } y = e^t - te^t - e^{3t}$$