Partial differential equation of first order

Let z = f(x, y) i.e. x and y are independent variable and z is dependent variable. That is two independent variable (x, y) and one dependent variable z. $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$

Order : Order of partial differential equation means order of highest derivative involved in the equation.
Degree : Degree of partial differential equation means power of highest order derivative involved in the equation.
Linear P. D. E. → If z and all its derivatives appears in the first degree and are not multiplied together.

Formation of P. D. E.

(a) By Eliminating Arbitrary Constant.

Exp. 1 Eliminate arbitrary 'a' and 'b' from z = ax + by and form a P. D. E. **Ans.** z = px + qy
Exp. 2 Form P. D. e. corresponding to z = (x + a) (y + b) here 'a' and 'b' are arbitrary constant. **Ans.** z = p.q
Exp. 3 Find P. D. E. by eliminating 'a' and 'b' from $z = (x^2 + a^2)(y^2 + b^2)$.

Solution : $p = 2x(y^2 + b^2)$, $q = 2y(x^2 + a^2)$, $y^2 + b^2 = \frac{p}{2x}$ => $x^2 + a^2 = \frac{q}{2y}$ Hence $z = \frac{p}{2x}.\frac{q}{2y}$ => 4xyz = pq Ans

Exp. 4 Eliminate arbitrary constant 'a' and 'b' from the equation $z = ae^{bx} \sin by$ and form the partial difference eqn.

Solution : $z = ae^{bx} \sin by$, $p = a \sin by\, e^{bx} b$, $q = ae^{bx} b \cos by$ Hence $z = \frac{p}{b}$

(b) By Elimination of Arbitrary function.

Exp. 1 Eliminate the arbitrary function *f* from $z = yf\left(\frac{y}{x}\right)$ and form P. D. E. **Ans.** pq + qy = z

Exp. 3 Obtain P. D. E. from z = f(sin x + cos y). **Ans.** p sin y + q cos x = 0

Exp. 4 Eliminate f from $z = f(x^2 - y^2)$.

Solution : $p = f'(x^2 - y^2).2x$, $q = f'(x^2 \text{ I } y^2)(-2y)$, $\frac{p}{q} = \frac{x}{-y}$ Hence py + qx = 0

Exp. 5 z = f(x + iy) + g(x – iy).
Solution : p = f'(x + iy) + g'(x – iy) , r = f''(x + iy) + g''(x – iy) , q = f'(x + iy) i + g'(x – iy)(-i) Hence r + t = 0

Solution of partial differential equation

linear and non-linear partial differential equation of First order -

Standard Form – 1
If equation is of type f(p, q) = 0 means equation have only terms of p and q, then z = ax + by + c ……(1)
Where a & b are constant and they are related as (a, b) = 0 , b = ϕ(a) Hence from (1) z = ax + ϕ(a)y + c

Exp. 1 $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$

Solution : $p^2 + q^2 = 1$, Its complete integral is given by; z = ax + by + c ……(1)
here a & b are related as $a^2 + b^2 = 1$ [Putting in $p^2 + q^2 = 1$, we get $a^2 + b^2 = 1$] & $b = \pm\sqrt{1-a^2}$
now putting value of b in equation (1) $z = ax \pm \sqrt{1-a^2}\, y + c$

Standard Form – 2

If equation is of type [z = px + qy + f(pq)] then its complete integral is, [z = ax + by + f(a, b)].

Exp. 1 $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial z}\right)^3}$

Solution : $z = px + qy + \sqrt{1+p^2+q^3}$ then its complete integral is $z = ax + by + \sqrt{1+a^2+b^3}$

Exp. 2 $z = px + qy + \sqrt{\sin^{-1}(p+q)}$ **Ans.** $ax + by + \sqrt{\sin^{-1}(a+b)}$

Exp. 3 $(z - px - qy) = \tan^{-1}(p^2+q^2) =>$ **Ans.** $z = (px+qy) + \tan^{-1}(p^2+q^2)$ i.e. $z = (ax+by) + \tan^{-1}(a^2+b^2)$

Standard Form – 3

f(p, q, z) = 0 equation having terms of p, q and z i.e., do not have terms of x and y. i.e. f(p, q, z) = 0

Let X = x + ay, its solution is z = f(X) i.e. z = f(x + ay) , $p = \frac{\partial z}{\partial x} \Rightarrow p = \frac{dz}{dX}.\frac{\partial X}{\partial x}$ & $q = \frac{dz}{dX}.\frac{\partial X}{\partial y}$

And in the given equation we substitute the value of $p = \frac{dz}{dx}$ & $q = a\frac{dz}{dx}$ and we get the differential equation of first degree and first order which can be solve by method of variable separable .

Exp. 1 z = pq

Solution Here equation have p, q & z Hence, X = x + ay And put $p = \frac{dz}{dX}$ & $q = a\frac{dz}{dX}$, $z = \frac{dz}{dX}.a\frac{dz}{dX}$

$$z = a.\left(\frac{dz}{dX}\right)^2 => \frac{dz}{dX} = \sqrt{\frac{z}{a}} \ , \int\frac{dz}{\sqrt{z}} = \int\frac{1}{\sqrt{a}}dX => z^{1/2} = \frac{1}{2}\left[\frac{1}{\sqrt{a}}(x+ay)+c\right] \quad \because [X = x+ay]$$

Exp. 2 p(1 + q) = qz **Ans.** $x + ay = a\log(az-1)+c$

Exp. 3 $q^2 = z^2 p^2 (1 - p^2)$ **Ans.** $z^2 = (x+ay+c)^2 + a^2$

Standard Form – 4 f(p, q, x, y) = 0 , This form does not contain terms of z. i.e. f(x, p) = φ(y , q) = a

First we assume f(x, p) = a & find the value of p & then we assume φ (y , q) = a & find the value of q. then we use, dz = pdx + qdy , We integrate it & solve it .

Exp. 1 $\sqrt{p} + \sqrt{q} = 2x$

Solution : $\sqrt{p} - 2x = -\sqrt{q} = a$ Now $\sqrt{p} - 2x = a$ => $\sqrt{p} = 2x + a$ => $p = (2x+a)^2$ and

$\sqrt{q} = a => q = a^2$ now dz = p dx + q dy , $\int dz = \int(2x+a)^2 dx + \int a^2 dy$ Hence $z = \frac{(2x+a)^3}{3} + a^2y + c$

Exp. 2 $p^2 + q^2 = x + y$

Solution : $p^2 - x = q^2 - y = a$ Now $p^2 - x^2 = a$ and $q^2 - y^2 = a$ i.e. $p = \sqrt{x+a}$ & $q = \sqrt{y+a}$

∴ dz = p dx + q dy Hence $\int dz = \int\sqrt{x+a}dx + \int\sqrt{y+a}dy$ i.e. $z = \frac{2}{3}\left[(x+a)^{3/2} + (y+a)^{3/2}\right] + c$

Reducible to Standard Form –

Exp. 1 $x^2p^2 + y^2q^2 = z^2$

Solution : $x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 = z^2 => \frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1$ [Adjusting x, y & z] => $\left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2 = 1$

Let, $\frac{\partial z}{z} = dZ$ and $\frac{\partial x}{x} = dX$ and $\frac{\partial y}{y} = dY$ Putting all the values in above equation ∴ $P^2 + Q^2 = 1$ [standard form I]

Complete integral Z = aX + bY + C Here, $a^2 + b^2 = 1 => b = \pm\sqrt{1-a^2}$ ∴ $Z = aX \pm \sqrt{1-a^2}Y + C$ ……(1)

now integrating $\int \frac{\partial z}{z} = \int \partial Z$, $\int \frac{\partial x}{x} = \int \partial X$, $\int \frac{\partial y}{y} = \int \partial Y$ => log z = Z, log x = X, log y = Y

now again putting values in equation (1) $\log z = a \log x \pm \sqrt{1-a^2} \log y + C$

Lagrange's method of solving the linear differential equation of first order

Equation $Pp + Qq = R$ is called Lagrange's equation here P, Q, R are f(x, y, z).

Working Rule :

Step I. Firstly we put the given partial differential equation in the standard form i.e., in the form given by

$$Pp + Qq = R \qquad \text{……(1)}$$

Step II. We write down the Lagrange's auxiliary equations for (1):

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{……(2)}$$

Step III. Now we find two independent integrals of auxiliary equations (2), say, u = a and v = b.

Step IV. Lastly the general solution of (1) is given by in any one of the following three equivalent forms:

$$\phi(u,v) = 0, u = \phi(v) \text{ or } v = \phi(u)$$

Exp. 1 $\frac{y^2 zp}{x} + zxy = y^2$

Solution. The given equation can be written as $y^2zp + x^2q = xy^2$

Thus subsidiary equations are $\frac{dx}{y^2 z} = \frac{dy}{zx^2} = \frac{dz}{xy^2}$

Taking the first two members, we have

$x^2\,dx = y^2\,dy$ $\therefore x^3 - y^3 = c_1$

Again taking the first and thirt members, we have

$x\,dx = z\,dz$ $\therefore x^2 - z^2 = c_2$

$\therefore$ The general solution is $f(x^3 - y^3, x^2 - y^2) = 0$

Exp. 2 $y^2 p - xyq = x(z - 2y)$. **[RGPV June 2003]**

Solution. The Lagrange's subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad \text{……(1)}$$

Taking first two fractions of (1), we have

$2x\,dx + 2y\,dy = 0$

$\therefore$ $x^2 + y^2 = c_1$ ……(2)

Now, taking last two fractions of (1), we have

$$\frac{dz}{dy} = -\frac{z-2y}{y} \Rightarrow \frac{dz}{dy} + \frac{1}{y}z = 2 \qquad \text{……(3)}$$

which is a linear differential equation in z, therefore its

I.F. $e^{\int\left(\frac{1}{y}\right)dy = e^{\log y} = y}$

So the solution of equation (3) is $zy - y^2 = c_2$ ……(4)

Hence the required general solution of the given equation is

$\phi(x^2 + y^2, zy - y^2) = 0$ where ϕ is an arbitrary function.

Exp. 3 Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ **[RGPV Dec. 02, 03, Jan. 07]**

Solution. The subsidiary equation are

$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$ Taking the multipliers 1, -1, 0; 0, 1, -1 and -1, 0, 1, we get

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)} \quad \ldots\ldots(1)$$

Taking first two members of (1), we get $\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$

Integrating, we have Log (x – y) = log (y – z) + log c_1 $\Rightarrow$ $\frac{x-y}{y-z}=c_1$

Now taking last two members and integrating, we get $\frac{z-x}{y-z}=c_2$

$\therefore$ The required general solution of the given equation is $\phi\left(\frac{x-y}{y-z},\frac{z-x}{y-z}\right)=0$

Exp. 4 Solve $(y^2+z^2-x^2)p-2xyq+2zx=0$ **or** $(x^2-y^2-z^2)p+2xyq=2xz$ **[RGPV June 04, Jan 06, Jun 14]**

Solution. The given equation may be written as $(y^2+z^2-x^2)p-2xyq=-2zx$

$\therefore$ The Lagrange's auxiliary equations are $\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2zx}$ ……(1)

Taking last two fractions of (1), we get $\frac{dy}{y}=\frac{dz}{z}\Rightarrow \log y=\log z+\log c_1 \Rightarrow \frac{y}{z}=c_1$ ……(2)

Now using x, y, z as multipliers, each fraction of (1)

$$\frac{x\,dx+y\,dy+z\,dz}{-x(y^2+z^2+x^2)}=\frac{dz}{-2zx}\Rightarrow\frac{2(x\,dx+y\,dy+z\,dz)}{x^2+y^2+z^2}=\frac{dz}{z}$$

Integrating, $\log(x^2+y^2+z^2)=\log z+\log c_2$ => $x^2+y^2+z^2=c_2 z$ ……(3)

Hence the general solution of the given equation is $x^2+y^2+z^2=z\phi\left(\frac{y}{z}\right)$

Charpit's Method

(General Method for solving partial differential equation with Two independent variable)

we use Charpit's subsidiary equation $\frac{dx}{\frac{-\partial f}{\partial p}}=\frac{dy}{\frac{-\partial f}{\partial y}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$

by solving it we get value of p and q then use formula dz = p dx + q dy on integrating it we get solution .

Exp. 1 Solve by Charpit's method px +qy = pq **[RGPV Jan 2007 , 13]**

Solution : Let $f(x,y,z,p,q)\equiv px+qy-pq=0$ ……(1) $\therefore \frac{\partial f}{\partial x}=p,\frac{\partial f}{\partial y}=q,\frac{\partial f}{\partial z}=0$, $\frac{\partial f}{\partial p}=x-q,\frac{\partial f}{\partial q}=y-p$.

Hence putting values in the charpit's auxiliary equations, namely

we get $\frac{dp}{p}=\frac{dq}{q}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dx}{q-x}=\frac{dy}{p-y}=\frac{dF}{0}$ Taking 1st and 2nd ratios, we get $\frac{dp}{p}=\frac{dq}{q}$ ….(2)

Integrating, log p = log q + log a => p = aq Substituting the above value of p in (1), we get

$q=\frac{ax+y}{a}$ ……(3) $\therefore$ from (2) and (3), p = ax + y……(4) & solution is $2az=(ax+y)^2+b$

Exp. 2 Solve by Charpit's method $z=px+qy+p^2+q^2$ **Ans.** $z=ax+by+a^2+b^2$

.

Exp. 3 Solve by Charpit's method $(p^2+q^2)y=qz$ **Ans.** $z^2=a^2y^2+(ax+b)^2$

Linear homogeneous p.d. equation with constant coefficient

Example 1 Solve the equation $4(r - s) + t = 16 \log (x + 2y)$

Solution We have $(4D^2 - 4DD' + D'^2)z = 16 \log (x + 2y)$

We can easily write the C.F. as $z = \phi_1(2y + x) + x\phi_2(2y + x)$

to get P.I. we proceed as follows $z = \dfrac{1}{4D^2 - 4DD' + D'^2}.16\log(x+2y)$ Here $f(a, b) = f(1, 2) = 0$.

Hence differentiating $F(D, D') = 4D^2 - 4DD' + D'^2$ w.r.t. D and multiplying the expression by a, we have

$z = x.\dfrac{1}{8D - 4D'}16\log(x+2y)$ Again performing same $z = x^2.\dfrac{1}{8}.16\log(x+2y) = 2x^2\log(x+2y)$.

Hence the complete solution is $z = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2 \log (x + 2y)$.

General Method This method is used when $F(DD')z$ is not equal to $f(ax + by)$ that is $F(DD') = f(x, y)$ that is any function of x and y.

Consider the equation $(D - m D')z = f(x, y)$. Then $P.I. = \dfrac{1}{D - mD'} f(x,y) = \int f(x, c - mx)dx$

Where c is to be replaced by y + mx after integration, since $y = c - mx$ or $c = y + mx$

Example 2 Solve $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

Solution The C.F. can be easily obtained as $z = \phi_1(y + 2x) + \phi_2(y - x)$.

To get P.I., we proceed as follows $z = \dfrac{1}{(D - DD' - 2D'^2)}(y-1)e^x = \dfrac{1}{(D+D')(D-2D')}(y-1)e^x$

$= \dfrac{1}{(D+D')}\int (c - 2x - 1)e^x\,dx = \dfrac{1}{(D+D')}(c-2x+1)e^x = \dfrac{1}{(D+D')}(y+1)e^x$ on replacing c by y + 2x.

Hence the complete solution is $z = \phi_1(y + 2x) + \phi_2(y - x) + ye^x$.

Example 3 Solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y$. $\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y + x)$

Solution The given equation is $(D + D')(D - D')z = \tan x \tan y(\tan^2 x - \tan^2 y) = \tan x \tan y(\sec^2 x - \sec^2 y)$.

$$\text{P.I.} = \frac{1}{(D+D')(D-D')}\tan x\tan y(\sec^2 x - \sec^2 y) = \frac{1}{D+D'}.\left[\int \tan x\tan(c-x)\{\sec^2 x - \sec^2(c-x)\}\,dx\right] (\because c - x = y)$$

$$= \frac{1}{D+D'}.\left[\int \tan x\tan(c-x)\sec^2 x\,dx - \int \tan x\tan(c-x)\sec^2(c-x)\,dx\right]$$

$$= \frac{1}{D+D'}.\left[\frac{\tan^2 x}{2}\tan(c-x) + \frac{1}{2}\int \tan^2 x\sec^2(c-x)\,dx + \frac{1}{2}\tan x\tan^2(c-x) - \frac{1}{2}\int \tan^2(c-x)\sec^2 x\,dx\right]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\tan(c-x) + \tan x\tan^2(c-x) + \int (\sec^2 x - \sec^2(c-x)\,dx\right]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\tan(c-x) + \tan x\tan^2(c-x) + \tan x + \tan(c-x)\right]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x.\tan y + \tan x\tan^2 y + \tan x + \tan y\right] \quad (\because y = c - x)$$

$$= \frac{1}{2(D+D')}\left[\tan y\sec^2 x + \tan x\sec^2 y\right] = \frac{1}{2}\int\left[\tan(k+x)\sec^2 x + \tan x\sec^2(k+x)\right]dx, \quad \text{where } k + x = y$$

$$= \frac{1}{2}\int\left[\frac{d}{dx}\{\tan x\tan(k+x)\}\right]dx = \frac{1}{2}\tan x\tan(k+x) = \frac{1}{2}\tan x\tan y \quad (\because k + x = y).$$

hence the complete solution is $z = \text{C.F.} + \text{P.I.} = \phi_1(y - x) + \phi_2(y + x) + \dfrac{1}{2}\tan x \tan y$.

One dimensional Wave Equation

An elastic string of length '*l*' which is stretched and then fixed at its two ends 'O' and A. Transverse defln y(x, t) at any point x at any time 't' when no external force act on it is given by the solution of P.D.E. $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Example 1 A string is stretched between the fixed points (0, 0) and (*l*, 0) and released at rest from position

$y = A\sin\frac{\pi x}{l}$. Find y(x, t).

Solution Here the boundary conditions are

y = (0, t) = y(*l*, t) = 0

and the initial conditions are

$y(x,0) = A\sin\frac{\pi x}{l}$ and $\frac{\partial y}{\partial t} = 0$ when t = 0

as discussed earlier, we have

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

where $A_n = \frac{2}{l}\int_0^l A\sin\frac{\pi x}{l}.\sin\frac{n\pi x}{l}\,dx$

$= \frac{2A}{l}\int_0^l \sin\frac{\pi x}{l}.\sin\frac{n\pi x}{l}\,dx$ $\int_0^l \sin mx.\sin nx\,dx = 0$ if m ≠ n

The above integral vanishes for all values of n except when n = 1. Then

$$A_1 = \frac{2A}{l}\int_0^l \sin^2\frac{\pi x}{l}\,dx$$

$$= \frac{A}{l}\int_0^l\left(1-\cos\frac{2\pi x}{l}\right)dx = \frac{A}{l}\left[x - \frac{l}{2\pi}\sin\frac{2\pi x}{l}\right]_0^l = A$$

Hence the particular solution of the wave equation under given conditions is

$y(x,t) = A\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}$. From equation (3)

Example 2 A tightly stretched string with fixed end points x = 0, and x = l is initially in a position given by $y = y_0\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement.

Solution The vibrations of the string are governed by one dimensional wave equation, given by

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \ldots\ldots(1)$$

The boundary and initial conditions are

y(0, t) = 0, y(*l*, t) = 0

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x) = 0$ $y(x,0) = f(x) = y_0\sin^3\left(\frac{\pi x}{l}\right)$

Therefore

$$y(x,t) = \sum_{n=1}^{\infty} A_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \quad \ldots\ldots(2)$$

where $A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\int_0^l y_0 \sin^3\left(\frac{\pi x}{l}\right)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2y_0}{l}\int_0^l \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)\sin\frac{n\pi x}{l}dx \qquad [\because \sin 3\theta = 3\sin\theta$$

$$= \frac{y_0}{2l}\left[\int_0^l 3\sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}dx - \int_0^l \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}dx\right]$$

clearly A_n vanishes for all n expect when n = 1 for first integral and n = 3 for second integral.

now $A_1 = \frac{3y_0}{2l}\int_0^l \sin^2\frac{\pi x}{l}dx = \frac{3y_0}{4l}\int_0^l\left(1-\cos\frac{2\pi x}{l}\right)dx = \frac{3y_0}{4}$

$$A_3 = -\frac{y_0}{2l}\int_0^l \sin^2\frac{3\pi x}{l}dx = \frac{y_0}{4l}\int_0^l\left(1-\cos\frac{6\pi x}{l}\right)dx = \frac{y_0}{4}$$

Thus the solution (2) becomes

$$y(x,t) = A_1\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} + A_3\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}$$

$$= \frac{y_0}{4}\left[3\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right]$$

Example 3 Find the solution of the wave equation **[RGPV Dec. 2012]**

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

such that y = p_o cos pt (p_o is a constant) when x = l and y = 0 when x = 0

Solution The given wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{……(1)}$$

the solution of (1) is given by

y(x, t) = (A cos kx + B sin kx)(C cos kct + D sin kct) ……(2)

Putting y = 0 when x = 0 in (2), we get A = 0 on putting in equation (2), we get

y(x, t) = B sin kx (C cos kct + D sin kct)

or y(x, t) = BC sin kx cos kct + BD sin kx sin kct ……(3)

Again putting y = p_0 cos pt when x = l in (3), we get

P_0 cos pt = (BC sin kl) cos kct + (BD sin kl) sin kct ……(4)

Equating terms of cos and sin on both side of (4), we obtain

p_o = BC sin kl or BC = p_o cosec kl

0 = BD sin kl, it gives D = 0 since B and sin kl both can not be zero,

and p = kc ⇒ k = p/c (equating angles).

Substituting values in (3), the requires solution is given by

$$y(x,t) = p_o \operatorname{cosec}\left(\frac{pl}{c}\right)\sin\frac{p}{c}x\cos pt \qquad [\because k = p/c]$$

Heat Equation

Let the temperature of the bar at any time t at a point x distance from the origin be u (x, t)

Then the equation of one dimensional heat flow is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$. **[RGPV Sept. 2009 , 12]**

Example 1 A rod of length "*l*" with insulated sides in initially at a temperature $u_o(x)$. Its ends are suddenly coded at 0^oC and are kept at that temperature prove that temperature function u(x, t) is given by $u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-} \frac{c^2\pi^2 n^2}{l^2} t$ where b_n is determine from the equation.

$b_n = \frac{2}{l}\int_0^l u_o(x) \sin \frac{n\pi x}{l} dx$.

Solution As we know that solution of heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given by $u = (A\cos kx + B\sin kx)e^{-K^2C^2t}$

Since the ends x = 0, x = l, are collected to 0^oC and kept at 0^oC so. u(0, t) = 0 , u(l, t) = 0 for All 't' initially u(x, 0) = $u_o(x)$ is initial condition on putting their condition. $0 = Ae^{-K^2C^2t}$, $\boxed{A = 0}$

Now $u = B\sin kx\, e^{-K^2C^2t}$, $0 = B\sin l\, e^{-K^2C^2t}$, sin kl = 0 = sin nπ n = 1, 2, 3,…. , kl = nπ , $K = \frac{n\pi}{l}$

Most general solution given by sum of all $\boxed{u = \sum_{n=1}^{\infty} \text{bn} \sin \frac{n\pi x}{l} e^{-} \frac{n^2\pi^2 c^2}{l^2} t}$

But initial be at t = 0, Temperature is $u_o(x)$ $u_o(x) = \sum_{n=1}^{\infty} \text{bn} \sin \frac{n\pi x}{l}$ ∴ $\text{bn} = \frac{2}{l}\int_0^l u_o(x) \sin \frac{n\pi x}{l} dx$

Example 2 Find the solution of $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ for which u(0, t) = u (l, t) = 0, u(x, 0) = sin $\frac{\pi x}{l}$ by method of variables separable.

Solution :

$$\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t} \qquad \ldots\ldots(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \qquad \ldots\ldots(2)$$

on comparing (1) and (2) we get $h^2 = \frac{1}{c^2}$

Thus solution of (1) is

$$u = (c_2 \cos px + c_3 \sin px) c_1 e^{\frac{p^2 t}{h^2}} \qquad \ldots\ldots(3)$$

on putting x = 0, u = 0 in (3) we get

$$0 = c_1 c_2 e^{\frac{p^2 t^2}{h^2}} \qquad 0 = c_1 \neq 0, \quad \therefore c_2 = 0$$

(3) is reduced to $u = c_3 \sin px c_1 e^{\frac{p^2 t}{h^2}}$ ……(4)

on putting x = l and u = 0 in (4) we get

$$0 = c_3 \sin pl c_1 e^{\frac{p^2 t}{h^2}} \qquad c_s \neq 0, c_1 \neq 0$$