Unit – Vector Calculus , Topic – Stokes Theorem

Line integral

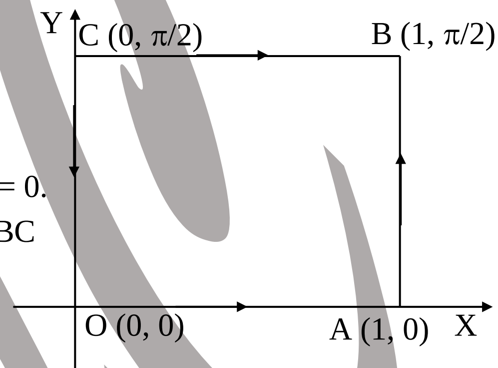
Q.1) Find the circulation of F along the curve C, where F = e^x sin y i + e^x cos y j, and C is the rectangle whose vertices are (0, 0), (1, 0), (1, π/2), (0, π/2).

Solution : Here r = xi + yj => dr = dxi + dyj

F.dr = (e^x sin yi + e^x cos yj).(dxi + dyj)

= e^x sin y dx + e^x cos y dy.

Draw the rectangle C as in the figure. Now along OA (i.e., on the x-axis), y = 0, dy = 0; along AB, x = 1, dx = 0; along BC, y = π/2, dy = 0; and along CO, x = 0, dx = 0.



So the circulation of F along C i.e., along the rectangle OABC

$$\int_C F.dr = \int_C (e^x \sin y\,dx + e^x \cos y\,dy)$$

$$= 0 + \int_0^{\pi/2} e\cos y\,dy + \int_1^0 e^x \sin\left(\frac{\pi}{2}\right)dx + \int_{\pi/2}^0 \cos y\,dy$$

$$= e[\sin y]_0^{\pi/2} + \left[e^x\right]_1^0 + [\sin y]_{\pi/2}^0 = e + (1-e) + (0-1) = 0$$

Q.2) Using the line integral compute the work done by the force F = (2y + 3) i + xz j + (yz – x) k when moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve x = $2t^2$, y = t, z = t^3.

Solution : Let C denote the arc of the given curve from the point A(0, 0, 0) to B(2, 1, 1). The given curve is x = $2t^2$, y = t, z = t^3. At A(0, 0, 0), t = 0 and B(2, 1, 1), t = 1. The required work done

$$\int_C F.dr = \int_C [(2y+3)i + xz\,j + (yz-x)k].(dx\,i + dy\,j + dz\,k)$$

$$= \int_C [(2y+3)\,dx + xz\,dy + (yz-x)\,dz] = \int_{t=0}^{1} [(2t+3).4t + 2t^5.1 + (t^4 - 2t^2).3t^2]\,dt$$

$$= \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt = \left[\frac{3t^7}{7} + \frac{t^6}{3} - \frac{6t^5}{5} + \frac{8t^3}{3} + 6t^2\right]_0^1 = \frac{3}{7} + \frac{1}{3} - \frac{6}{5} + \frac{8}{3} + 6 = \frac{3}{7} - \frac{6}{5} + 9 = \frac{288}{35} \text{ units.}$$

Q.3) Find the total work done in moving a particle in a force field given by F = 3xy i – 5z j + 10x k along the curve x = t^2 + 1, y = $2t^2$, z = t^3 from t = 1 to t = 2. **Ans. 303Unit**

Q.4) Find the work done in moving a particle in a force field F = $3x^2$i+ (2xz – y) j + zk along the line joining (0, 0, 0) to (2, 1, 3).

Solution : The equation of the straight line joining (0, 0, 0) to (2, 1, 3) are

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \quad \text{i.e.,} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{, say}$$

∴ x = 2t, y = t, z = 3t are parametric equations of the curve (here straight line) C. At the point (0,0,0), t = 0 and at the point (2, 1, 3), t = 1.

∴ Required work done $= \int_C F.dr = \int_C [3x^2\,i + (2xz-y)\,j + z\,k].[dx\,i + dy\,j + dz\,k]$

$$= \int_C [3x^2\,dx + (2xz-y)\,dy + z\,dz] = \int_0^1 [12t^2.2 + (12t^2 - t).1 + 3t.3]\,dt = \int_0^1 (36t^2 + 8t)\,dt = 16 \text{ units}$$

Stoke's Theorem

Statement : If S is the open, two sided surface bounded by a non-intersecting curve C then for the vector F having continuous derivatives $\int_C F.dr = \int_S (curl\ F).n\,dS$, where n is the unit normal vector at any point of S and C is traversed in the positive direction. The direction of C is called positive if an observer with his head in the direction of n, has the surface on the left.

Q.5) Verify Stoke's Theorem when $F = (2x-y)i - yz^2 j - y^2 zk$ where S is the upper half of the sphere $x^2+y^2+z^2 = i$ and C is the boundary.

Solution : Given $F = (2x-y)i - yz^2 j - y^2 zk$

Also we know

r = xi + yj + zk

$\therefore \quad F.dr = [(2x-y)i - yz^2 j - y^2 zk].[i\,dx + j\,dy + k\,dz]$

or $\quad F.dr = (2x-y)dx - yz^2 dy - y^2 z dz \qquad$ ……(i)

Also here C is the circle $x^2+y^2=1$; z = 0 i.e. a circle of unit radius on xy-plane.

$\therefore$ For C we have z = 0, dz = 0. Consequently from (i) we have

F . dr = (2x – y) dx $\qquad$ ……(ii)

Also on C we have x = cos ϕ and y = sin ϕ and ϕ varies from 0 to 2π.

$\therefore$ From (ii) we have

$$\int_C F.dr = \int_C (2x-y)\,dx = \int_{\phi=0}^{2\pi}(2\cos\phi - \sin\phi)(-\sin\phi)d\phi$$

$$= \int_{\phi=0}^{2\pi}(\sin^2\phi - \sin 2\phi)d\phi = 4\int_0^{\pi/2}\sin^2\phi\,d\phi + \left[\frac{1}{2}\cos 2\phi\right]_0^{2\pi} = 4.\frac{1}{2}.\frac{1}{2}\pi + \frac{1}{2}[1-1] = \pi$$

Hence $\int_C F.dr = \pi \qquad$ ……(iii)

Also the direction cosines of the line OP are

sin θ cos ϕ, sin θ sin ϕ, cos θ

$\therefore \quad$ n = (sin θ cos ϕ)i + (sin θ sin ϕ)j + (cos θ)k,

where n is the unit outward drawn normal vector at P.

$$\text{Again curl F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = i\left[\frac{\partial}{\partial y}(-y^2 z) - \frac{\partial}{\partial z}(-yz^2)\right]$$

$$+j\left[\frac{\partial}{\partial z}(2x-y) - \frac{\partial}{\partial x}(-y^2 z)\right] + k\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right] = 0i + 0j + k = k$$

$\therefore$ n . curl F = [(sin θ cos ϕ)I + (sin θ sin ϕ)j + (cos θ)k] . k = cos θ

$\therefore \quad \int_S n.curl\ F\ dS$

$$= \int_S \cos\theta\,dS \text{, where dS = sin θ dθ dϕ} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\frac{1}{2}\sin 2\theta\,d\theta\,d\phi = \frac{1}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}[\phi]_0^{2\pi}$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0\right]2\pi = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right).2\pi = \pi \qquad \text{……(iv)}$$

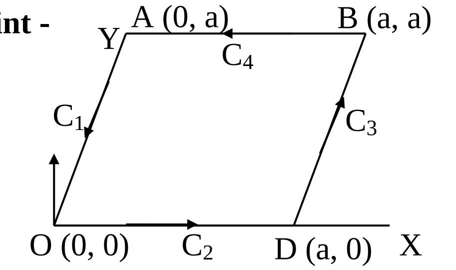
Hence from (iii) and (iv) we have

$$\int_C F.dr = \pi = \int_S n.curl\,F\,dS$$

which verifies Stoke's Theorem.

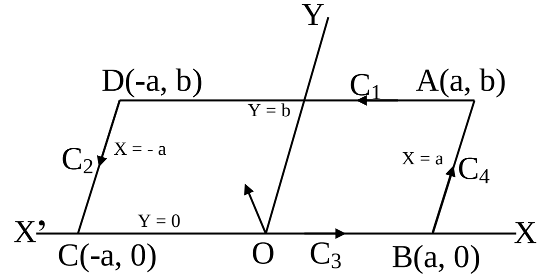
Q.6) Verify Stoke's Theorem when $F = x^2 i + xyj$, where C is the perimeter of the square in xy-plane whose sides are along the lines x = 0, y = 0, x = a, y = a.

Ans. $\frac{1}{2}a^3$ **Hint -**



Q.7) Verify Stoke's Theorem for $F = (x^2 + y^2)i - 2xyj$ taken round the rectangle bounded by $x = \pm a$, y = 0, y = b.

Solution : Here $F = (x^2 + y^2)i - 2xyj$ and dr = (dx)i + (dy)j



$\therefore F.dr = \left[(x^2 + y^2)i - 2xyj\right].\left[(dx)i + (dy)j\right]$

$= (x^2 + y^2)dx - 2xy\,dy$ ……(i)

Let C be the perimeter of the given rectangle and C_1, C_2, C_3 and C_4 be the lines from A to D, D to C, C to B and B to A respectively.

Then $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + \int_{C_3} F.dr + \int_{C_4} F.dr$ ……(ii)

Now for C_1 we have y = b, dy = 0 and x varies from a to -a, so from (i)

We get $\int_{C_1} F.dr = \int_{x=a}^{-a} (x^2 + b^2)dx = \left[\frac{1}{3}x^3 + b^2 x\right]_a^{-a} = -\left(\frac{2}{3}a^3 + 2b^2 a\right)$ ……(iii)

For C_2 we have x = -a, dx = 0 and y varies from b to 0, so from (i)

We get $\int_{C_2} F.dr = \int_{y=b}^{0} (2ay)\,dy = -2a\left(\frac{1}{2}y^2\right)_0^b = -ab^2$ ……(iv)

For C_3 we have y = 0, dy = 0 and x varies form - a to a, so form (i)

We get $\int_{C_3} F.dr = \int_{x=-a}^{a} x^2\,dx = \left(\frac{1}{3}x^3\right)_{-a}^{a} = \frac{2}{3}a^3$ ……(v)

For C_4 we have x = a, dx = 0 and y varies from 0 to b, so from (i)

We get $\int_{C_4} F.dr = -\int_{y=0}^{b} 2ay\,dy = -\left(ay^2\right)_0^b = -ab^2$ ……(vi)

$\therefore$ From (ii), (iii), (iv), (v) and (vi) we have

$\int_C F.dr = -\left(\frac{2}{3}a^3 + 2b^2 a\right) - ab^2 + \frac{2}{3}a^3 - ab^2 = -4ab^2$ ……(vii)

Again as is evident from the figure, the unit outward normal vector n = k.

Also curl $F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = i[0] - j[0] + k\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right] = -4y\,k$

$\therefore$ n . curl F = k . (- 4y k) = - 4y

Also dS = dx dy **(See figure)**

$\therefore \int_S n.curl\ F\,dS = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dx\,dy = -4\int_{-a}^{a}\left(\frac{1}{2}y^2\right)_0^b dx = -2b^2\left(x\right)_{-a}^{a} = -4ab^2$ ……(vii)

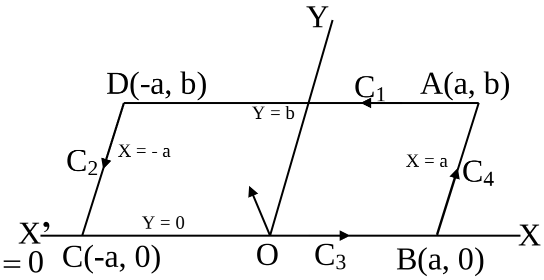
$\therefore$ From (vii) and (viii) we get

$\int_C F.dr = \int_S n.curl\ F\,dS$

Q.8) Verify Stoke's Theorem for the vector field defined by $F = (x^2 - y^2)i + 2xyj$ in the rectangular region in the xy-plane bounded by the lines x = 0, x = a , y = 0 and y = b.

Ans. $2ab^2$

Q.9) Evaluate by Stoke's Theorem $\int_C (y\,z\,dx + z\,x\,dy + x\,y\,dz)$, where C is the curve $x^2 + y^2 = 1,\ z = y^2$



Solution : Here $\int_C (y\,z\,dx + z\,x\,dy + x\,y\,dz)$

$$= \int_C (y\,z\,i + z\,x\,j + x\,y\,k).(i\,dx + j\,dy + k\,dz) = \int_C (y\,z\,i + z\,x\,j + x\,y\,k).dr$$

$\therefore$ "F" = y z **i** + z x **j** + x y **k**

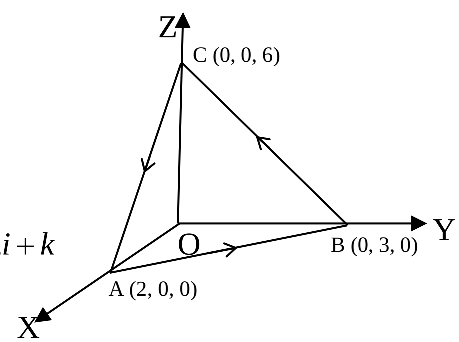
$$\therefore \text{ curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x-x)i - (y-y)j + (z-z)k = 0$$

$\therefore$ By Stoke's theorem, the given integral

$= \int_S n.curl\ F\,dS$

= 0, since curl F = 0

Q.10) Apply Stoke's Theorem to evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).



Solution : Let $\vec{F} = (x+y)i + (2x-z)j + (y+z)k$

Then the given integral reduces to

$$\int_C \vec{F}.d\,\vec{r} = \iint_S \text{curl } \vec{F}.\hat{n}\,dS$$

where S is the surface of the triangle ABC.

$$\text{Now curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-y & y+z \end{vmatrix} = i(1+1) - j(0) + k(2-1) = 2i + k$$

The equation of the plane ABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \text{ or } 3x + 2y + z = 6.$$

Then unit vector normal to the plane ABC is given by

$$\hat{n} = \frac{3i + 2j + k}{\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3i + 2j + k)$$

$$\therefore \quad \text{Curl } \vec{F}.\hat{n} = (2i+k).\frac{1}{\sqrt{14}}(3i + 2j + k) = \frac{7}{\sqrt{14}}$$

$$\text{Hence } \iint_S (\text{curl } \vec{F}.\hat{n})\,dS = \iint_R (\text{curl } \vec{F}\ \hat{n})\frac{dx\,dy}{n.k} = \iint_R \left(\frac{7}{\sqrt{14}}\right)\frac{dx\,dy}{1/\sqrt{14}} = 7\iint_R dx\,dy,$$

Where R is the region bounded by triangle OAB. Now

$$7\iint_R dx\,dy = 7\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} dx\,dy = 7\int_0^2 \left[\frac{6-3x}{2}\right]dx = \frac{7}{2}\left[6x - \frac{3x^2}{2}\right]_0^2 = 21$$